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Mean-Variance Portfolio Selection with Reference Dependent Preferences*

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Abstract. We study S-shaped utility maximization for the standard portfolio selection problem with one risky and one risk-free asset. We derive a mean-variance criterium of choice, which preserves reference dependence and the reflection effect. Subsequently we study diversification possibilities and obtain the demand for the risky asset. We close the paper with an alternative interpretation of the criterium in terms of target-based decision making.

Keywords: portfolio selection, S-shaped utility, prospect theory, reference point, mean-variance analysis, demand for the risky asset, target-based decisions.

JEL classification: D81, G11.

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1. Introduction

The mean-variance approach of Markowitz (1952) has been the most prominent tool for portfolio selection. Since its introduction, it was used as a major framework for financial decision making when faced with risky alternatives. Its power lies in its simplicity due to the dependence of the investment decision on only two moments of the return distribution function of the assets. Moreover, the mean-variance approach is equivalent to the expected utility (EU) theory in case the utility function of an agent is quadratic or the returns of the assets are normally distributed.[1]

As a special case of the original Markowitz approach a criterium of choice, which is linear in the mean and the variance of the risky alternative, can be derived. The additional assumption necessary for its derivation is strict concavity and the constant absolute risk aversion property of the decision maker's utility function, the normality of the returns distribution being retained. This criterium became widely employed due to its intuitive and easy to apply analytically tractable form.

However, the criterium has also been criticized. The reason is that it is characterized by the utility properties that make the agent's decisions be compatible with the expected utility concept, which itself is disputed and is constantly attacked by many due to the inconsistences it exhibits in real problems of choice under risk and uncertainty. Two choice anomalies usually thought of when one speaks about the EU are the Allais paradox and the Ellsberg paradox that serve as examples of the independence axiom failure.

The prospect theory (PT) of Kahneman and Tversky (1979) was designed to overcome some of the anomalies that the EU framework suffers from. One of its main features is the existence of a reference point with respect to which decision makers code outcomes into gains and losses. Therefore, no more referring to the final outcomes as in the EU. This reference dependence appears in light of a special form of the utility function, which is taken to be S-shaped rather than strictly concave as in the EU framework. Although the PT is highly attractive for its descriptive properties, the lack of a closed form criterium of choice based on a mean-variance functional detracts from its appeal.

We propose such a criterium in this work. Our approach allows to enrich the EU with reference dependence and the reflection effect (opposite preferences for positive and negative outcomes), two important and often documented features of human decisions. The choice criterium obtained is mean-variance compatible and is in a intuitively clear analytically tractable form. It can be used in many choice frameworks and specifically in portfolio selection, which we are going to discuss extensively below. The criterium can be interpreted as being based on a relaxed form of the PT with loss aversion of the value function being relaxed and no probability distortion being employed.

The availability of such a criterium makes it possible to derive the demand for the risky asset, which could be used in various models of financial markets and many

[1] This extends to the class of elliptical distributions (to which normal distribution belongs).

other settings. Obtaining the demand is a difficult task when the preferences of the agent admit reference dependence in the form of the PT value function mostly due to its non-differentiability at the inflection (reference) point. Several studies attempted to derive the demand but for all we know not a single effort was successful in obtaining it in a closed form.

A noticeable example is the work of Hwang and Satchell (2005) who focus on parameterizations of the PT value function both using analytical tools and calibration methods. They speculate about the conditions under which one can obtain the demand for the risky asset for the general form of the return distribution. Davies and Satchell (2005) enrich the former model with nonlinear probability weighting mechanism and concentrate on continuous prospects. However, in both cases the demand is not found in a closed form and the authors have to rely on empirical computations to obtain it. Gomes (2005) explores a two-period optimal portfolio allocation problem of a loss-averse investor with an extended version of the PT value function. The value function is modelled concave over the domain of large losses so as to limit the amount of risk the agent can take. Also, the reference point can react to changes in the risky asset prices, thus, its dynamics is modelled explicitly. The author notices that the demand for the risky asset must be obtained numerically unless the return distribution of the risky asset is a two-state discrete one. Anderson (2004) studies the properties of the indifference curves of a loss-averse investor assuming normally distributed return of the risky asset. Numerical methods are used for the analysis as the analytical treatment is unmanageable.

The attempts to relax some of the prospect theory assumptions, for example, making the value function piece-wise linear as in e.g. Barberis et al. (2001) and Anderson (2004), or dropping the normality of the risky asset return and switching to another distributions, e.g. Gamma distribution as in Hwang and Satchell (2005) and Davies and Satchell (2005), unfortunately do not lead to analytically tractable solutions. In all such cases it is necessary to implement numerical tools for the demand derivation.

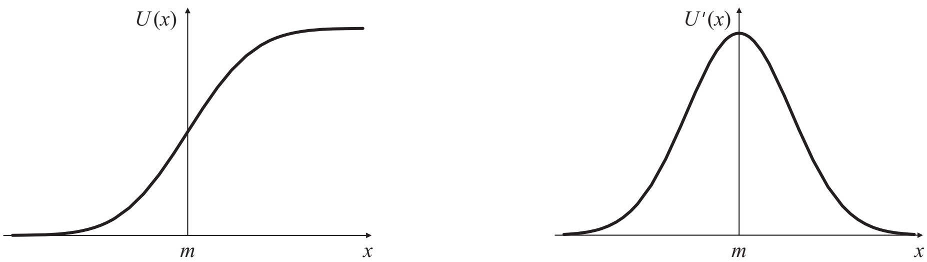
The structure of the paper is as follows. In Section 2 we focus on portfolio selection problem with normal utility function. We derive a mean-variance criterium of choice under risk and obtain the demand for the risky asset. Section 3 contains the discussion of the risk attitude of the agent endowed with an S-shaped utility. Section 4 is devoted to the graphical analysis of the criterium and its conformity with the well-known safety first criteria. Section 5 expounds a target-based interpretation of our approach, while Section 6 gives concluding remarks of the work and points out to future research. Auxiliary results are available in the appendix.

2. Portfolio selection with S-shaped utility

Consider a real-valued S-shaped function $U(x)$ that possesses the following properties: it is strictly increasing, continuous, once differentiable, bounded[2] over its possibly unbounded support, and symmetric around its unique inflection point m. For simplicity we assume that the function is twice differentiable so that

$$U''(x) \begin{cases} \geq 0 & \text{if } x \leq m, \\ < 0 & \text{if } x > m, \end{cases} \tag{1}$$

where $m = \text{argmax}\{U'(x)\}$. Essentially this means that the first derivative function is unimodal. The graphs of $U(x)$ and its first derivative are depicted in Figure 1. This notion of a symmetric S-shaped function is used throughout the text.



(a) An S-shaped function

(b) The first derivative

Figure 1: An S-shaped function and its first derivative.

When interpreted as a von Neumann-Morgenstern utility function representing preferences over risky lotteries, the above stated properties of $U(x)$ imply that an agent would be strictly risk averse over the outcomes greater than m, and strictly risk seeking over the outcomes less than m. At the inflection point the agent is locally risk neutral.

If the inflection point m is treated as a reference point of $U(x)$, as we shall call it from now on, then we are in the PT framework with the utility function defined over changes of the dependent variable, for example wealth, rather than over its absolute values. However, the symmetric curvature of $U(x)$ suggests that the agent would have risk attitude of the same extent towards gains and losses of equal magnitude. Apparently such a utility function underfulfils the properties of the PT since loss aversion is not modelled.

For analytical tractability of the model we turn now to the special case of the general symmetric utility function of wealth, namely, to the normal utility function.

Consider now the standard portfolio selection problem. Suppose that a myopic agent with the initial wealth $W_t > 0$ maximizes the expected value of her utility of the future wealth $U(\boldsymbol{W}_{t+1})$ by choosing every period of time an amount α_t to invest

[2]The range of $U(x)$ is defined on a bounded set, i.e. $\lim_{x\to-\infty} U(x) = a$ and $\lim_{x\to\infty} U(x) = b$, where a and b are, correspondingly, the lower and the upper bounds of the function.

in the risky asset, allocating $(W_t - \alpha_t)$ to the risk-free asset.[3] The assets, thus, are traded in discrete time. The risk-free asset is perfectly elastically supplied and pays a constant gross return $R_f = 1 + r_f$, while the risky asset pays an uncertain return $\boldsymbol{R}_{t+1}$. Let the ex post income at date $t+1$ after the rate of return on the risky asset is realized be given by[4]

$$\boldsymbol{W}_{t+1} = \alpha_t \boldsymbol{R}_{t+1} + (W_t - \alpha_t)R_f = \alpha_t(\boldsymbol{R}_{t+1} - R_f) + W_t R_f. \tag{2}$$

The choice of the component α_t in (2) generates a portfolio, which can be described as a lottery over monetary outcomes. We do not impose any restrictions on α_t as short selling is allowed and riskless lending and borrowing is possible.

We assume that the random return of the risky asset is normally distributed, so we have $\boldsymbol{W}_{t+1} \sim N(\mu, \sigma)$, where $\mu = \mathrm{E}_t[\boldsymbol{W}_{t+1}]$ and $\sigma = \sqrt{\mathrm{Var}_t[\boldsymbol{W}_{t+1}]}$. All the expectations henceforth will be taken with respect to the distribution of $\boldsymbol{R}_{t+1}$.

Our aim is to maximize the expected value of the S-shaped utility function $U(\boldsymbol{W}_{t+1})$, that is to $\max_{\alpha_t}\{G(\alpha_t)\} = \max_{\alpha_t}\{\mathrm{E}_t[U(\alpha_t(\boldsymbol{R}_{t+1} - R_f) + W_t R_f)]\}$. The following Proposition offers a way to obtain the result in a closed form. The proof is presented in Appendix.

Proposition 1. *For any S-shaped utility function $U(\boldsymbol{W}_{t+1})$ the following holds true:*

$$\mathrm{argmax}_{\alpha_t}\{\mathrm{E}_t[U(\boldsymbol{W}_{t+1})]\} = \mathrm{argmax}_{\alpha_t}\{\frac{\mu - m}{\sqrt{\sigma^2 + s^2}}\}, \tag{3}$$

where m is the inflection point of the function $U(\boldsymbol{W}_{t+1})$, and s is its measure of dispersion around m.

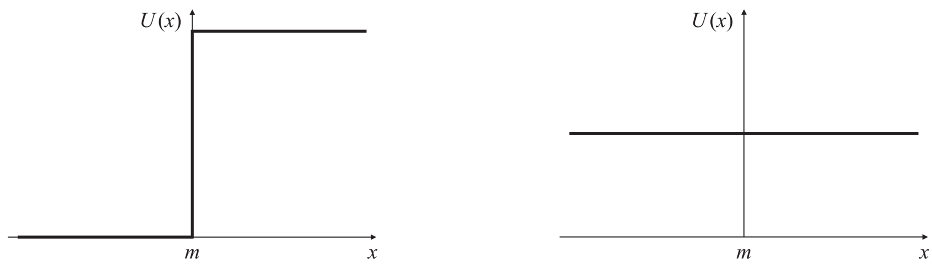
The parameter m can be considered as the location parameter of $U(\boldsymbol{W}_{t+1})$ besides its interpretation as the reference point, which was discussed earlier. Analogously the parameter s can be regarded as the scale parameter. In fact, if $s = 0$, then the utility becomes a step function as depicted in Figure 2a. If, conversely, $s \to \infty$, then the function becomes a straight line, stretched out on the whole domain as shown in Figure 2b.

Given that $\mu = \alpha_t\{\mathrm{E}_t[\boldsymbol{R}_{t+1}] - R_f\} + W_t R_f$ and $\sigma^2 = \alpha_t^2 \mathrm{Var}_t[\boldsymbol{R}_{t+1}]$, the maximization problem for the normal utility with normally distributed risky asset return becomes

$$\max_{\alpha_t}\left\{\frac{\alpha_t\{\mathrm{E}_t[\boldsymbol{R}_{t+1}] - R_f\} + W_t R_f - m}{\sqrt{\alpha_t^2 \mathrm{Var}_t[\boldsymbol{R}_{t+1}] + s^2}}\right\}. \tag{4}$$

[3] Bold face type is used to denote random variables at date $t+1$.

[4] Assuming that the risky asset pays stochastic dividend y_t at the beginning of each trading period t and denoting its ex-dividend price per-share in period t by p_t, the gross return on the risky asset becomes $\boldsymbol{R}_{t+1} = \frac{\boldsymbol{p}_{t+1} + \boldsymbol{y}_{t+1}}{p_t}$. Furthermore, substituting $z_t p_t$ for α_t and expanding the gross return in (2) gives the following wealth dynamics in terms of the future price and the dividend of the risky asset: $\boldsymbol{W}_{t+1} = p_t z_t \frac{\boldsymbol{p}_{t+1} + \boldsymbol{y}_{t+1}}{p_t} + (W_t - p_t z_t)R_f = W_t R_f + z_t(\boldsymbol{p}_{t+1} + \boldsymbol{y}_{t+1} - p_t R_f)$, where z_t is the number of shares of the risky asset purchased at date t.



(a) $s = 0$ (b) $s \to \infty$

Figure 2: An S-shaped function for extreme values of s.

From the first order condition the only critical point is

$$\alpha_t^* = \frac{s^2(\mathrm{E}_t[\boldsymbol{R}_{t+1}] - R_f)}{(W_t R_f - m)\mathrm{Var}_t[\boldsymbol{R}_{t+1}]}. \tag{5}$$

The second order condition at the critical point holds if $W_t R_f > m$, indeed:

$$G''(\alpha_t^*) = -\frac{s^4(\mathrm{E}_t[\boldsymbol{R}_{t+1}] - R_f)^2 + s^2(W_t R_f - m)^2\mathrm{Var}_t[\boldsymbol{R}_{t+1}]}{(W_t R_f - m)([\alpha_t^*]^2\mathrm{Var}_t[\boldsymbol{R}_{t+1}] + s^2)^{5/2}} < 0. \tag{6}$$

Hence, the critical point is a local maximum. Furthermore, taking into account that this critical point is unique, it is also the global maximum of the function. The interior solution to the maximization problem is strictly positive if $\mathrm{E}_t[\boldsymbol{R}_{t+1}] > R_f$ since

$$G'(0) = \frac{\mathrm{E}_t[\boldsymbol{R}_{t+1}] - R_f}{s}. \tag{7}$$

Therefore, (5) gives the demand for the risky asset.

The choice criterium in (3) depends only on the first two moments of the risky asset return distribution, which makes it fully compatible with the mean-variance analysis. We term it the RMV criterium, where the first letter stands for "reference dependence".

We sum everything up by stating that (5) is the amount of the optimal investment in the risky asset if and only if $W_t R_f > m$ and $\mathrm{E}_t[\boldsymbol{R}_{t+1}] > R_f$. This implies that the agent will always construct a diversified portfolio when the risk premium is strictly positive and when the agent's reference point is strictly less than risk-free investment.

By putting $m = 0$ we simultaneously set the reference point of the agent equal zero and the optimal solution becomes[5]

$$\alpha_t^* = \frac{s^2(\mathrm{E}_t[\boldsymbol{R}_{t+1}] - R_f)}{W_t R_f \mathrm{Var}_t[\boldsymbol{R}_{t+1}]}. \tag{8}$$

Unfortunately, it is not possible to prove similar diversification result for a general symmetric S-shaped utility function as it could be, for instance, done for a strictly concave utility (see e.g. LeRoy and Werner (2001) for a proof of the diversification theorem for the case of a strictly risk averse investor). As a matter of fact, one could pick a symmetric S-shaped function for which the expected value would have an infinite number of local extrema, the arctangent being an appropriate choice.[6] This finding is demonstrated in Appendix.

3. S-shaped utility and attitude to risk

From the agent's behavior perspective the reference point condition for the existence of the optimal solution can be explained as follows. Suppose that the risk premium is positive ($\mathrm{E}_t[\boldsymbol{R}_{t+1}] > R_f$). The agent is risk averse if the riskless investment is greater than the reference point ($W_t R_f > m$) and she is risk prone if it is less than the reference point. Hence, given that the risky asset dominates the risk-free one, she will invest all her wealth into the risky asset when risk prone ($\alpha_t \to +\infty$) and only some limited positive amount of her wealth when risk averse ($\alpha_t > 0$), avoiding any big loss in case the risky asset underperforms the risk-free asset in the next period: not too much is still good.

If the risk premium is negative, the situation changes. This time the risky asset is dominated by the risk-free one, therefore, it is never optimal to invest any positive amount in the former one. Moreover, since short selling of α_t is possible, the agent will short sell some limited amount of the risky asset ($\alpha_t < 0$) whenever the risk-free investment alone guarantees overshooting the reference point ($W_t R_f > m$). This way the agent increases her profit by investing more wealth than she has in the more lucrative asset. At the same time her risk aversion does not let her engage in short selling too much because the risky asset may still outperform the risk-free one, which could lead to a possibly large loan repaying cost. If, however, the risk-free investment is not enough for attaining the reference level ($W_t R_f < m$), the agent will infinitely short-sell the risky asset ($\alpha_t \to -\infty$) under the effect of her risk propensity.

We note that although the reference point is often set equal to the risk-free investment (see e.g. Barberis et al. (2001)), in our set-up this would lead to an

[5] We can restate the result in (8) in terms of the future price $\boldsymbol{p}_{t+1}$ and the dividend $\boldsymbol{y}_{t+1}$ of the risky asset by recalling Footnote 4. In fact, if we denote $\alpha_t^* = p_t z_t^*$, where z_t^* is the optimal number of shares of the risky asset bought, then the demand for the risky asset becomes

$$z_t^* = \frac{s^2(\mathrm{E}_t[\boldsymbol{p}_{t+1} + \boldsymbol{y}_{t+1}] - p_t R_f)}{W_t R_f \mathrm{Var}_t[\boldsymbol{p}_{t+1} + \boldsymbol{y}_{t+1}]}.$$

[6] Igor Bykadorov is gratefully acknowledged for making this example.

infinite amount of risky investment if the risk premium were positive, and to an infinite amount of short selling if the risk premium were negative, so diversification would never be achieved.

The absolute risk aversion function of the normal utility looks as follows:

$$r(x) = \frac{x - m}{s}, \tag{9}$$

where the parameter m stands for the reference point. From (9) it is clear that besides being a scale parameter s affects the risk attitude of the agent: the fraction grows in absolute value over the whole domain with the decrease of s. In particular, risk aversion of the agent (risk attitude over favorable outcomes $x > m$) and her risk propensity (risk attitude over unfavorable outcomes $x < m$) increase as s rises, and they decrease as s falls. That is, the agent becomes less sensitive to risk when s is high and more sensitive to risk when s is low.

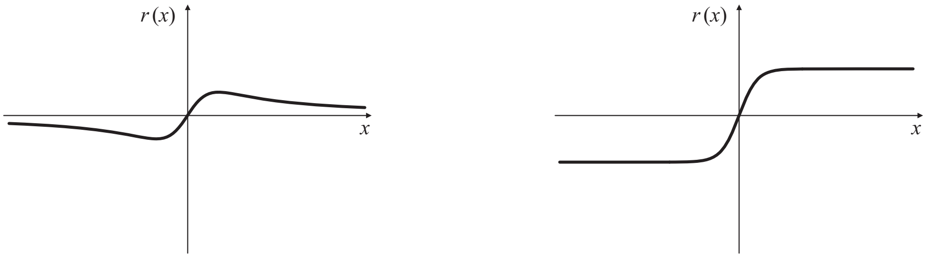
The risk attitude measure in (9) converges to its highest possible value at $s = 0$ when the utility becomes a step function with the greatest curvature (see Figure 2a); it attains its lowest possible value at the limiting case of $s \to \infty$ when the utility function becomes a straight line (see Figure 2b).

Unfortunately, the risk aversion function for the normal utility does not seem completely realistic. In fact, one can conclude that the demand for the risky asset exhibits increasing absolute risk aversion property since the absolute sum invested in the risky asset is decreasing with the initial wealth. According to a widely accepted viewpoint, risk aversion should decrease with the increase of wealth (at least for large values of wealth), while in the case of normal utility it increases. Even though the increase of s reduces this affect, the positive slope of the risk aversion function remains.

We provide an intuition on why the agent exhibits increasing absolute risk aversion property when she is endowed with the normal utility function. In essence, the agent's attitude towards risk is conditional on how large her initial endowment of wealth is. Since the probability of falling short the reference level is always positive due to the distributional assumptions, the larger the reference level in absolute terms is the more risk averse the agent becomes. This is intuitively clear as people usually tend to fear losses more when they have more to lose (see also the discussion of dual risk attitude in Section 2 of DellaVigna and LiCalzi (2001)). This intuition may fail though for a large level of wealth.

We considered several alternative choices for a symmetric S-shaped utility function but were unable to derive a mean-variance criterium in a closed form for any except for the normal utility. We examined the arctangent ($\arctan(x) + \frac{\pi}{2}$), the exponential arctangent ($\arctan(e^x)$), the hyperbolic tangent ($\tanh(x) + 1$), and the logistic function ($\frac{1}{1+e^{-x}}$) besides the normal c.d.f.

The arctangent possesses the most realistic absolute risk aversion function, which decreases as wealth increases, while retaining an appealing switching hump-shaped section around zero (see Figure 3a). Somewhat less attractive shape of the risk



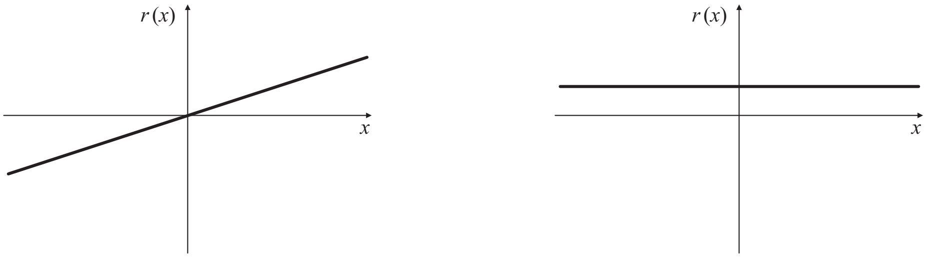
(a) Arctangent function

(b) Logistic curve

Figure 3: Risk aversion function for arctan(x) and logistic utilities.

aversion function is related to the exponential arctangent, the hyperbolic tangent and the logistic curve (see Figure 3b). The risk aversion function graph of the normal c.d.f. hits the mark only on a bounded interval around zero, which, however, compares almost exactly with the above mentioned four examples (see Figure 4a).

It is clear that all the four described utility functions (except for the normal) give rise to more realistic risk attitude of the agent than the commonly employed negative exponential utility, which leads to the constant absolute risk aversion property (see Figure 4b).



(a) Normal c.d.f.

(b) Exponential function

Figure 4: Risk aversion function for normal and exponential utilities.

Although closed form solutions to the four appealing alternatives to the normal c.d.f. are not found, an approximation to the arctangent as the most realistic case is derived in Gerasymchuk (2007). This approximation can be used in a more computationally oriented and less analytically tractable models of portfolio selection.

4. Graphical analysis

In this section we carry out a graphical interpretation for the diversification possibilities of the agent who relies on the RMV criterium in (3). We also discuss a special case when the scale parameter s of the utility function is set to zero, which leads to the safety first criterium originally introduced in Roy (1952) and which implies no diversification. Our set-up of the previous sections is now extended to the multiasset case, and the initial wealth is normalized to unity for readability. Note that the extension of the optimal solution in (5) to the multiple assets case is straightforward and is not displayed here.

We remind that the agent who maximizes the expected value of his normal utility function under the assumption of normally distributed risky asset return equivalently solves

$$\max\{\frac{\mu - m}{\sqrt{\sigma^2 + s^2}}\}, \tag{10}$$

where μ and the σ are the mean and the standard deviation of the portfolio of the agent.

All the equally desirable portfolios would have the same value of the agent's expected utility function or, alternatively, they would all have the same value of this ratio. Denote such a value T so that all such portfolios could be described by

$$\frac{\mu - m}{\sqrt{\sigma^2 + s^2}} = T. \tag{11}$$

We can derive the indifference curves of the agent in the mean-standard deviation space by rearranging the above formula and setting the mean of the portfolio μ be dependent variable, while making the standard deviation σ be the independent one:

$$\mu = m + T\sqrt{\sigma^2 + s^2}. \tag{12}$$

We can see immediately that the indifference curves are strictly increasing and convex as long as T takes only positive values,[7] which we assume here because of the first order stochastic dominance implications. The curves intersect the vertical axis at the point $\mu = m + Ts$, which shifts upward as T, the level of satisfaction, increases. Also they become steeper with the increase of T. Besides, the curves have an inclined asymptote[8]

$$\mu = m + T\sigma. \tag{13}$$

The portfolio that maximizes the RMV criterium is the tangency point of an indifference curve and the efficient frontier in the mean-standard deviation diagram.

We refer here to Levy and Levy (2004) who showed that the efficient set under the prospect theory-like preferences almost coincide with the efficient frontier under the standard mean-variance preferences. In particular, it can be proved that under very reasonable assumptions[9] one can obtain the PT efficient set by excluding a

[7] The first and the second order conditions are, correspondingly, $\mu'_\sigma = T\sigma/\sqrt{\sigma^2 + s^2} > 0$ and $\mu''_\sigma = Ts^2/\sqrt{\sigma^2 + s^2} > 0$, so the indifference curves are strictly increasing and convex as T, σ, s are all strictly positive.

[8] The asymptote of the function $f(\sigma) = m + T\sqrt{\sigma^2 + s^2}$ is given by the equation $\mu = k\sigma + b$, where k (its slope) can be found from $\lim_{\sigma \to +\infty} f(\sigma)/\sigma$ and b (its intersection with the vertical axis) can be obtained by taking the limit $\lim_{\sigma \to +\infty} [f(\sigma) - k\sigma])$. Implementing the calculations leads to the equation of the asymptote given in (13).

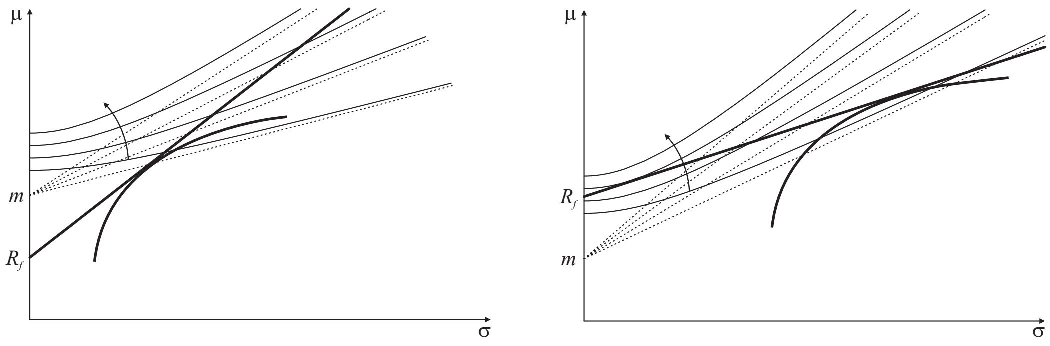
[9] The returns of the assets must be normally distributed, the assets must not be perfectly correlated, and no restrictions can be put on portfolio formation.

small segment to the left of the minimum variance portfolio from the mean-variance efficient frontier.[10]

In the presence of the riskless lending and borrowing the efficient frontier becomes a straight line connecting the point of the risk-free asset return on the vertical axis and the tangency portfolio. In terms of the two-asset standard portfolio problem its slope is given by $\frac{\mathrm{E}_t[\boldsymbol{R}_{t+1}]-R_f}{\sigma}$. We consider here only the case of strictly positive risk premium ($\mathrm{E}_t[\boldsymbol{R}_{t+1}] > R_f$), thus, the slope is positive too.

In Figure 5a we can see that the optimal portfolio would be the one with infinite demand for the risky asset if the intersection of the asymptotes of the indifference curves with the vertical axis lies above or on a par with the risk-free rate of return. This happens because the curves would shift in the counter-clockwise direction to the north-west corner and their tangency point with the efficient frontier would tend to infinity. If, however, the asymptotes of the indifference curves cross the vertical axis below the risk-free asset return, then there will be a curve with a finite tangency point (see Figure 5b).

In the figure the dashed lines represent asymptotes, the solid lines denote indifference curves, while the bold line stands for the efficient frontier. The topmost solid curve represents the indifference curve with the highest satisfaction level, while the downmost one, accordingly, denotes the lowest satisfaction indifference curve. It is clear that the topmost indifference curve in Figure 5b is not feasible as it does not intersect the efficient set and is depicted for the representation clarity only to emphasize the tangency (optimal) curve preceding it.



(a) No diversification exists ($\alpha_t \to \infty$)

(b) Diversification exists ($\alpha_t > 0$)

Figure 5: Investment alternatives for $s > 0$.

Therefore, diversification exists whenever the asymptote's intercept is strictly less than the risk-free rate of return, that is, whenever $R_f > m$. This result was shown in Section 2 but now it is accompanied with an intuitive graphical interpretation.

A special case of the RMV criterium occurs when s is set to zero so that (10)

[10]See Theorem 1 in Levy and Levy (2004). The result was obtained by making use of the Prospect Stochastic Dominance rule originally introduced in Levy and Wiener (1998).

becomes

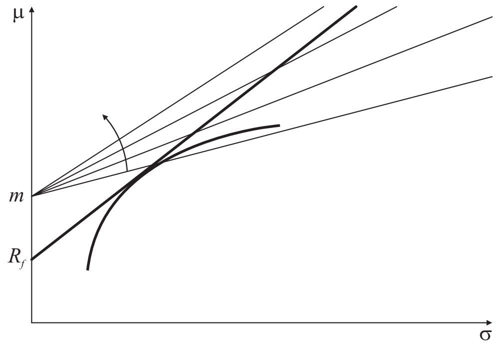
$$\max\{\frac{\mu - m}{\sigma}\} \tag{14}$$

and the RMV criterium turns into the safety first criterium of Roy (1952), which was further modified by Telser (1955) and Kataoka (1963).

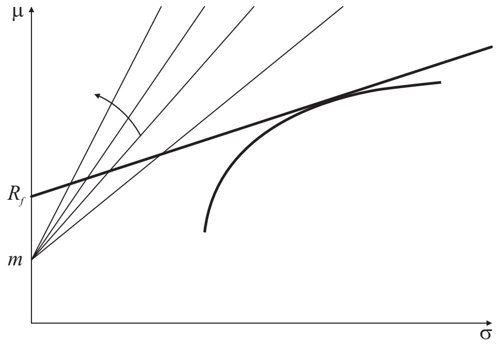
The indifference curves equation is now a straight line

$$\mu = m + T\sigma, \tag{15}$$

with m being its intercept with the vertical axis and T being its slope.



(a) No diversification exists ($\alpha_t \to \infty$)



(b) No diversification exists ($\alpha_t = 0$)

Figure 6: Investment alternatives for $s = 0$.

Hence, the indifference curve (or line) with the highest level of satisfaction would be the one with the highest slope. The indifference curves would become steeper as T increases without shifting their intercept upward (see Figure 6). There will be no diversification in this case because no matter where the intercepts of the indifference curves are the demand for the risky asset will always be either zero or infinite. In fact, from Figure 6 it is clear that if $m > R_f$, then the indifference curve with the highest level of satisfaction (the topmost one) would intersect the efficient frontier at infinity. If $R_f > m$, then the highest satisfaction indifference curve would be the one that coincides with the vertical axis and, thus, the agent would invest all the initial wealth into the risk-free asset. If $m = R_f$ then there would be infinitely many solutions. Similar graphical analysis for the safety first criteria was carried out by Pyle and Turnovsky (1970) and Elton et al. (2003). As in Figure 5, the solid lines denote indifference curves, while the bold line represent the efficient frontier.

Safety first criterium has a very intuitive and appealing interpretation, namely that the agent when making a decision is not maximizing her expected utility but rather sets some reference level below which she does not want to fall short. Thus, the agent maximizes the probability of achieving at least her reference level. This interpretation originates from the following criterium:

$$\max\{P(R_p \geq m)\}, \tag{16}$$

which is the original Roy's criterium that can be shown to be equivalent to (14) under the assumption of normal distribution of the portfolio's return R_p. In the above formula m denotes the reference level and P stands for the probability operator.

In the next section we will show that equivalent interpretation applies also to the RMV criterium and that, in fact, m can be treated as the mean of some target, which is not deterministic (as in the Roy's original model) but stochastic. Correspondingly, the standard deviation of such target would be s, thus, providing a simple link between the RMV and safety first criteria: the target becomes deterministic as soon as s is set equal zero.

5. Target-based interpretation

The RMV criterium can be given an interpretation equivalent to the intuition provided by Roy for his safety first criterium, which is quite different from the common notion of the expected utility maximization. It is discussed extensively in Castagnoli and LiCalzi (1996) for the case of decision making under risk and in Bordley and LiCalzi (2000) for decision making under uncertainty.

This interpretation restates the treatment of the expected utility by means of a purely probabilistic language so that the decisions of the agent are no more based on the notion of a cardinal utility: the agent starts thinking in terms of probabilities instead.

In fact, as shown in Proof of Proposition 1, the maximization of the expected normal utility of a normally distributed random variable $\boldsymbol{W}_{t+1}$ is equivalent to the maximization of the probability of that variable to outperform another, stochastically independent, random variable $\boldsymbol{V}$. The latter one can be interpreted as some (stochastic) target wealth of the agent. The mean of $\boldsymbol{V}$ is the reference level below which all the outcomes would be considered as losses, and, accordingly, the outcomes equal or above it would be perceived by the agent as gains.

Castagnoli and LiCalzi (1996) show that the ranking of two monetary lotteries $\boldsymbol{X}$ and $\boldsymbol{Y}$ with probability distributions F and G in accordance with the EU concept is equivalent to the procedure that ranks them based upon the probability of their exceeding a stochastically independent target $\boldsymbol{V}$ (given that utility functions $U(\boldsymbol{X})$ and $U(\boldsymbol{Y})$ can be viewed as c.d.f.'s after appropriate normalization):

$$\boldsymbol{X} \succeq \boldsymbol{Y} \text{ iff } \mathrm{E}[U(\boldsymbol{X})] \geq \mathrm{E}[U(\boldsymbol{Y})], \tag{17}$$

where

$$\mathrm{E}[U(\boldsymbol{X})] = \int_{-\infty}^{+\infty} U_{\boldsymbol{V}}(x)dF(x) = \int_{-\infty}^{+\infty} P(\boldsymbol{V} \leq x)dF(x) = P(\boldsymbol{X} \geq \boldsymbol{V}) \tag{18}$$

and

$$\mathrm{E}[U(\boldsymbol{Y})] = \int_{-\infty}^{+\infty} U_{\boldsymbol{V}}(y)dG(y) = \int_{-\infty}^{+\infty} P(\boldsymbol{V} \leq y)dG(y) = P(\boldsymbol{Y} \geq \boldsymbol{V}) \tag{19}$$

$$\Rightarrow \boldsymbol{X} \succeq \boldsymbol{Y} \text{ iff } P(\boldsymbol{X} \geq \boldsymbol{V}) \geq P(\boldsymbol{Y} \geq \boldsymbol{V}). \tag{20}$$

Note that the functionals of both the expected utility and target-based frameworks are linear in probabilities. Equivalence of the two procedures means that the axiomatic foundation of one of them is suitable also for the other. Therefore, when maximizing the expected utility the agent acts as if she is maximizing the probability of not falling short the target, and vice versa.

We can see that the two ranking procedures described above coincide when $\mathrm{E}[U(\boldsymbol{X})] = P(\boldsymbol{X} \geq \boldsymbol{V})$ or $U(x) = P(x \geq \boldsymbol{V})$, that is, when the utility of a lottery is interpreted as a probability. Bordley and LiCalzi (2000) point out that such interpretation of a utility function is completely sensible. Indeed, if there exist two polar (the best and the worst) outcomes, then a utility function would be the probability p that makes the agent indifferent between obtaining for sure some intermediate outcome and a lottery yielding the best outcome with probability p and the worst outcome with probability $(1-p)$.

The stochastic nature of $\boldsymbol{V}$ can be interpreted as follows. Suppose that the agent is unable to decide which target is the most suitable to her. So the natural choice would be to pick a target, which is imperfectly known, that is, a target of the form $\boldsymbol{V} = d + \varepsilon$, where the first component of the right-hand part is deterministic reflecting the agent's admissible knowledge of her target, while the second term of the sum is a zero-mean error, thus, representing hesitation of the agent about what the right target should be. The agent, thus, makes an assessment of the distribution of $\boldsymbol{V}$ and considers the expected value of the probabilistic criterium over all possible reference levels:

$$P(\boldsymbol{X} \geq \boldsymbol{V}) = \int_{-\infty}^{+\infty} P(\boldsymbol{X} \geq v) dU(v). \tag{21}$$

Classical von Neumann-Morgenstern utility function $U(x)$ embodies the agent's uncertainty about the right target. Looking at its first derivative, which translates into a density function by means of probabilistic language, one could notice that a strictly concave $U(x)$ would imply that the agent assesses the target in a conservative (risk averse) way, namely, more probability is attached to the worst outcomes. If, instead, $U(x)$ is S-shaped and symmetric around its inflection point, then it becomes clear that the agent perceives the target symmetrically distributed around the reference level. Therefore, she becomes risk averse in the domain of gains (positive outcomes if the reference point is zero) and risk seeking in the domain of losses (negative outcomes).[11]

We can see from the preceding discussion that the target-based procedure presents a more intuitive treatment of the decision making process since it exploits an easily understood notion of probability rather than somewhat subtle concept of cardinal utility. Besides, the probabilistic language makes it more accessible how and why

[11] See Section 2 of Bordley and LiCalzi (2000) for a more detailed treatment of the risk attitude of the agent depending on the form of $U(x)$.

reference dependence works. However, given that both the target-based and the utility-based ranking procedures are equivalent, it is actually disputable which of the two be to some extent better. An interested reader is referred to the work of LiCalzi (1999) where a thorough comparison of the two procedures is presented.

We note that target-based approach recalls the satisficing principle discussed in Simon (1955), which is suitable for modelling bounded rationality, a framework of rational choice under some processing and cognitive limitations. The Satisficing approach states that the agent assesses some threshold above which all the outcomes are considered favorable and, instead of optimizing, simply picks the first action to satisfy such a constraint. Nevertheless, target-based framework remains practically optimizing as the probability of meeting the target is maximized.

6. Conclusion

In this paper a criterium of choice under risk is analyzed, which is based on a symmetric S-shaped utility function. The criterium is derived from the normal utility maximization and incorporates a desirable feature of reference dependence. The demand for the risky asset is obtained in an analytically tractable closed form, which can be applied to different frameworks, financial markets equilibrium modelling being one of the most prominent.

An analysis of the criterium including diversification possibilities is conducted and it is shown that the reference point must be strictly less than the full investment in the risk-free asset for diversification to exist. Moreover, it is demonstrated that if the target of a decision-maker is degenerate, as it is the case for the safety first criteria, then the agent does not diversify her portfolio.

As a future perspective a generalization of the criterium to the lottery-dependence framework of Becker and Sarin (1987) can be studied by discarding the condition of stochastic independence of the target. The relaxation of this condition leads to some form of dependence of the utility function on the statistical properties of the lottery. Exactly such a phenomenon is examined in the work of Becker and Sarin.

Appendix

Proof of Proposition 1. The result in (3) is due to DellaVigna and LiCalzi (2001). We repeat its derivation here.

An S-shaped utility $U(x)$ can be normalized by means of an appropriate positive affine transformation to take values from zero to unity. The range of the function is the interval $[0, 1]$, while its domain is $\Re$. All the other properties of an S-shaped function as defined above are retained. We draw the attention of the reader that any function satisfying these properties is also a cumulative distribution function (c.d.f.). We assume that this c.d.f. is a normal one.

Since the utility function can be viewed as a normal c.d.f., the maximization problem of the agent can be restated in the following way:

$$\max_{\alpha_t}\{\mathrm{E}_t[U(\boldsymbol{W}_{t+1})]\} = \max_{\alpha_t}\{\mathrm{E}_t[F_{\boldsymbol{V}}(\boldsymbol{W}_{t+1})]\}, \tag{22}$$

where $F_{\boldsymbol{V}}(\boldsymbol{W}_{t+1})$ stands for the c.d.f. of some random variable $\boldsymbol{V}$ stochastically independent of $\boldsymbol{W}_{t+1}$. Denote $\mathrm{E}[\boldsymbol{V}] = m$ and $\mathrm{Var}[\boldsymbol{V}] = s^2$.

Taking into account that $\boldsymbol{W}_{t+1} \sim N(\mu, \sigma)$ and $\boldsymbol{V} \sim N(m, s)$, the difference of these two random variables is also normally distributed:

$$(\boldsymbol{V} - \boldsymbol{W}_{t+1}) \sim N((m - \mu), (\sqrt{\sigma^2 + s^2})), \tag{23}$$

and it follows that

$$Z = \frac{(\boldsymbol{V} - \boldsymbol{W}_{t+1}) - (m - \mu)}{\sqrt{\sigma^2 + s^2}} \sim N(0, 1). \tag{24}$$

By making use of the fact that

$$\begin{aligned} \mathrm{E}_t[F_{\boldsymbol{V}}(\boldsymbol{W}_{t+1})] &= \int_{-\infty}^{+\infty} F_{\boldsymbol{V}}(w_{t+1}) f(w_{t+1}) dw_{t+1} \\ &= \int_{-\infty}^{+\infty} P(\boldsymbol{V} \leq w_{t+1}) f(w_{t+1}) dw_{t+1} = P(\boldsymbol{V} \leq \boldsymbol{W}_{t+1}) \end{aligned} \tag{25}$$

we obtain

$$P(\boldsymbol{V} - \boldsymbol{W}_{t+1} \leq 0) = P(Z \leq \frac{\mu - m}{\sqrt{\sigma^2 + s^2}}) = \Phi(\frac{\mu - m}{\sqrt{\sigma^2 + s^2}}), \tag{26}$$

where Φ is a standard normal c.d.f., which is a strictly increasing function, hence, (3) follows immediately.[12] $\square$

General S-shaped expected utility. We show here that the diversification result such as in (4) cannot be obtained for the general S-shaped utility function with continuously distributed asset returns.

Consider the arctangent function $\arctan(x) + \frac{\pi}{2}$ as an example of the S-shaped utility. In this case the expected utility of the future wealth takes the following form:

$$\begin{aligned} G(\alpha_t) = p_1 \cdot \arctan[\alpha_t(R_1 - R_f) + W_t R_f] + ... \\ + p_n \cdot \arctan[\alpha_t(R_n - R_f) + W_t R_f] + \frac{\pi}{2}, \end{aligned} \tag{27}$$

where $R_1, ..., R_n$ are the realizations of the random variable $\boldsymbol{R}_{t+1}$, and $p_1, ..., p_n$ are the corresponding probabilities of these realizations.

Consider now the simplest possible non-degenerate distribution of $\boldsymbol{R}_{t+1}$. Suppose that the random variable can take only two values: R_1 and R_2 with probability p and $(1-p)$ correspondingly. The first order condition is

$$\frac{p(R_1 - R_f)}{1 + [\alpha_t(R_1 - R_f + W_t R_f)]^2} + \frac{(1-p)(R_2 - R_f)}{1 + [\alpha_t(R_2 - R_f + W_t R_f)]^2} = 0. \tag{28}$$

After collecting terms it becomes clear that the degree of the resulting polynomial equals two. In general, its degree depends on the number of realizations k of $\boldsymbol{R}_{t+1}$ and is equal to $2(k-1)$. Even though we cannot guarantee that the number of real roots of the first order condition will equal the degree of the polynomial, it can certainly become extremely large when possible realizations increase. This is true also for symmetric distributions of $\boldsymbol{R}_{t+1}$. Consequently, the obtaining of the global maximum becomes practically unmanageable as the objective function is not unimodal and may have an infinite number of local extrema.

As an illustrative example of nonunimodality of the objective function consider the following simple discrete distribution: $R_1 = -100$, $p_1 = 1/8$; $R_2 = 50$, $p_2 = 2/8$; $R_3 = -1$, $p_3 = 3/8$; $R_4 = 20$, $p_4 = 2/8$. If the initial wealth and the risk-free return are taken to be $W_t = 1$ and $R_f = 2$, then the maximization of (27) with respect to α_t will reveal two local maxima.

[12]The derivation of the choice criterium in (3) can also be seen as resulting from a discrete choice set-up, namely Probit, within the random utility framework. See Manski and McFadden (1990) for further details.

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